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ON HARARY ENERGY OF GRAPHS

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Abstract: The Harary matrix of a connected graph G is defined as $H(G) = [a_{ij}]_{n\times n}$, where $a_{ij} = \dfrac{1}{d(v_i, v_j)}$; for v_i and v_j are non adjacent in G and $a_{ii} = 0$; for all $i, j = 1, 2, 3, \cdots, n$. The Harary energy of G is the sum of the absolute values of the eigenvalues of Harary matrix of G. In this paper, the Harary characteristic polynomial of $K_{m,n}$ and Harary energy of some graphs are investigated.

Keywords and Phrases: Eigenvalue, Graph Polynomial, Graph Energy.

2020 Mathematics Subject Classification: 05C50, 05C31, 05C76.

1. Introduction and Preliminaries

Let G be a simple, undirected and connected graph with vertex set $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. The *distance* between two vertices v_i and v_j is the length of shortest path between them; for all $1 \leq i, j \leq j$. The maximum distance between any pair of vertices is known as *diameter* of graph G. For standard terminology and notations in graph theory, rely upon West [14] while for any undefined term related to energy of graphs, refer to Gutman [6].

Definition 1.1. *The m-Shadow graph, $D_m(G)$ of a connected graph G is constructed by taking m copies of G say $G_1, G_2, ..., G_m$. Then Join each vertex u in*

G_i to the neighbors of the corresponding vertex v in G_j, $1 \le i, j \le m$.

Definition 1.2. *The extended m-Shadow graph, $D^*_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, ..., G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v and with v in G_j, $1 \le i, j \le m$.*

The *adjacency matrix* of a graph G is a symmetric matrix of order n defined as $A(G) = [a_{ij}]_{n\times n}$, where $a_{ij} = 1$, if v_i and v_j are adjacent in G and $a_{ij} = 0$, if v_i and v_j are non adjacent G. The *energy* $\mathcal{E}(G)$ of a graph G is the sum of absolute values of eigenvalues of adjacency matrix of a graph G with their multiplicity. The concept of energy was introduced by Gutman [5] in 1978 and further explored by [7, 12]. Some variants of graph energy like Distance Energy [10], Randić Energy [2, 13], Color Energy [1] and Laplacian Energy [8] are also available in literature.

The Harary matrix of a connected graph G is defined as $H(G) = [a_{ij}]_{n\times n}$, where $a_{ii} = 0$ and $a_{ij} = \dfrac{1}{d(v_i, v_j)}$, if $i \ne j$, for all $1 \le i, j \le n$. The Harary characteristic polynomial of a graph G is a characteristic polynomial of the Harary matrix, $H(G)$. The eigenvalues of the matrix $H(G)$ is known as Harary eigenvalues of G and the Harary spectrum of G is denoted as $spec_H(G)$. The Harary energy of a graph G is the sum of the absolute values of the Harary eigenvalues of G with their multiplicity $\left(i.e.\ \mathcal{E}_H(G) = \sum_{i=1}^{n} |h_i|\right)$, where h_i is the Harary eigenvalue of G for all $1 \le i \le n$. The concept of Harary energy was introduced by G̈unğor and Çevik [4] and further studied by [3, 11]. In the present paper, the Harary characteristic polynomial of $K_{m,n}$ and Harary energy of some graphs are obtained.

For our ready reference, some existing results are stated below.

Proposition 1.1. [9] *Let*

$$M = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

be a symmetric matrix. Then the spectrum of M is the union of spectrum of $A+B$ and $A-B$. i.e. $spec(M) = spec(A+B) \cup spec(A-B)$.

Proposition 1.2. [9] *Let $A, B, C, D \in \mathbb{R}^{n\times n}$ be matrices, Q be an invertible matrix and*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

then $detM = detD \cdot det\,(A - BD^{-1}C)$.

Proposition 1.3. [9] *Let A and $A-B$ be the invertible matrices then $(A-B)^{-1} = A^{-1} + \dfrac{1}{1+t}A^{-1}BA^{-1}$, where $t = trace(-BA^{-1})$.*

Proposition 1.4. [9] *Let $A = [a_{ij}]_{n\times n}$ be any matrix such that $a_{ii} = a$ and $a_{ij} = b$; for $i \neq j$ then $det A = (a + (n-1)b)(a-b)^{n-1}$.*

Proposition 1.5. [9] *If λ is an eigenvalue of the matrix $A = [a_{ij}]_{n\times n}$ with corresponding eigenvector x and μ is an eigenvalue of the matrix $B = [b_{ij}]_{m\times m}$ with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

2. Main Results

Theorem 2.1. *For $n \geq 2$, $\mathcal{E}_H(K_{n,n}) = 3n - 1$.*
Proof. Let $K_{n,n}$ be the complete bipartite graph with the bipartition $\{v_1, v_2, ..., v_n\} \cup \{u_1, u_2, ..., u_n\}$, where $n \geq 2$ be a positive integer then the Harary matrix of $K_{n,n}$ is defined as

$$H(K_{n,n}) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \\ \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \boldsymbol{u_3} \\ \vdots \\ \boldsymbol{u_n} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} & \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \cdots & \boldsymbol{u_n} \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow H(K_{n,n}) = \begin{bmatrix} A & B \\ B & A \end{bmatrix}_{2n\times 2n}$$

where,

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \end{bmatrix}_{n\times n} \quad \text{and } B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n} = J_n$$

By Proposition 1.3, $spec(H(K_{n,n})) = spec(A+B) \cup spec(A-B)$.

Now,

$$A+B = \begin{bmatrix} 1 & \frac{3}{2} & \frac{3}{2} & \cdots & \frac{3}{2} \\ \frac{3}{2} & 1 & \frac{3}{2} & \cdots & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 1 & \cdots & \frac{3}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \cdots & 1 \end{bmatrix}_{n\times n}$$

$$\Rightarrow det(xI_n - (A+B)) = \begin{vmatrix} x-1 & -\frac{3}{2} & -\frac{3}{2} & \cdots & -\frac{3}{2} \\ -\frac{3}{2} & x-1 & -\frac{3}{2} & \cdots & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} & x-1 & \cdots & -\frac{3}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} & \cdots & x-1 \end{vmatrix}$$

Now, using Proposition 1.6, we get

$$det(xI_n - (A+B)) = \left(x - 1 - (n-1)\frac{3}{2}\right) \cdot \left(x - 1 + \frac{3}{2}\right)^{n-1} = \left(x - \frac{3n-1}{2}\right) \cdot \left(x + \frac{1}{2}\right)^{n-1}$$

$$\Rightarrow spec(A+B) = \begin{pmatrix} \frac{3n-1}{2} & -\frac{1}{2} \\ 1 & n-1 \end{pmatrix}$$

Similarly,

$$A - B = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 & \cdots & -\frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & -1 \end{bmatrix}_{n\times n}$$

$$\Rightarrow det(xI_n - (A-B)) = \left(x + \frac{n+1}{2}\right) \cdot \left(x - \left(\frac{1}{2}\right)\right)^{n-1}$$

$$\Rightarrow spec(A-B) = \begin{pmatrix} -\frac{n+1}{2} & \frac{1}{2} \\ 1 & n-1 \end{pmatrix}$$

Therefore,

$spec(H(K_{n,n})) = spec(A+B) \cup spec(A-B)$

$$= \begin{pmatrix} \frac{3n-1}{2} & -\frac{1}{2} & -\frac{n+1}{2} & \frac{1}{2} \\ 1 & n-1 & 1 & n-1 \end{pmatrix} \qquad (1)$$

Hence,

$$\begin{aligned} \mathcal{E}_H(K_{n,n}) &= \left|\frac{3n-1}{2}\right| + \left|-\frac{n+1}{2}\right| + \left|-\frac{1}{2}(n-1)\right| + \left|\frac{1}{2}(n-1)\right| \\ &= \frac{3n-1}{2} + \frac{n+1}{2} + \frac{1}{2}(n-1) + \frac{1}{2}(n-1);\ \text{as } n \geq 3 \\ &= \frac{6n-2}{2} \\ &= 3n-1 \end{aligned}$$

$\Rightarrow \mathcal{E}_H(K_{n,n}) = 3n-1$

Theorem 2.2. *Let $m, n \geq 2$ be the positive integers such that $m \neq n$ then the*

Harary characteristic polynomial of the complete bipartite graph $K_{m,n}$ is

$$\varphi_H(K_{m,n};x) = \left(x - \frac{n-1}{2}\right)\left(x+\frac{1}{2}\right)^{m+n-2}\left(x - \frac{m-1}{2} - \frac{mn}{x} - \frac{mn(n-1)}{2x^2}\right)$$

Proof. Let $K_{m,n}$ be the complete bipartite graph with the bipartition $\{v_1, v_2, ..., v_m\} \cup \{u_1, u_2, ..., u_n\}$, where, $m, n \geq 2$ be the positive integers such that $m \neq n$ then the Harary matrix of $K_{m,n}$ is defined as

$$H(K_{m,n}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_{(m+n)\times(m+n)}$$

where, $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \end{bmatrix}_{m\times m}$ $\quad B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{m\times n}$

$C = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times m}$ and $D = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & 0 \end{bmatrix}_{n\times n}$

Now, using Proposition 1.4, the Harary characteristic polynomial of $K_{m,n}$ is

$$\begin{aligned} \varphi(K_{m,n};x) &= |xI_{m+n} - H(K_{m,n})| \\ &= \begin{vmatrix} xI_m - A & B \\ C & xI_n - D \end{vmatrix} \qquad (2) \\ &= det(xI_n - D) \cdot det((xI_m - A) - B(xI_n - D)^{-1}C) \end{aligned}$$

Now, using the Proposition 1.5, we have

$$(xI_n - D)^{-1} = (xI_n)^{-1} + \frac{1}{1+t}(xI_n)^{-1}D(xI_n)^{-1} = \frac{1}{x}I_n + \frac{1}{x^2}D;$$

where, $t = trace(-D(xI_n)^{-1}) = 0$.

So,

$$B\left(\frac{1}{x}I_n + \frac{1}{x^2}D\right)C$$

$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{m\times n} \begin{bmatrix} \frac{1}{x} & \frac{1}{2x^2} & \frac{1}{2x^2} & \cdots & \frac{1}{2x^2} \\ \frac{1}{2x^2} & \frac{1}{x} & \frac{1}{2x^2} & \cdots & \frac{1}{2x^2} \\ \frac{1}{2x^2} & \frac{1}{2x^2} & \frac{1}{x} & \cdots & \frac{1}{2x^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2x^2} & \frac{1}{2x^2} & \frac{1}{2x^2} & \cdots & \frac{1}{x} \end{bmatrix}_{n\times n} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times m}$$

$$= \left(\frac{1}{x} + \frac{n-1}{2x^2}\right) \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{m\times n} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times m}$$

$$= n\left(\frac{1}{x} + \frac{n-1}{2x^2}\right) \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{m\times m}$$

$$= n\left(\frac{1}{x} + \frac{n-1}{2x^2}\right) J_m$$

Moreover,

$$xI_m - A - B\left(\frac{1}{x}I_n + \frac{1}{2x^2}D\right)C$$

$$= \begin{bmatrix} x & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & x & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & x & \cdots & -\frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & x \end{bmatrix}_{m\times m} - n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) J_m$$

$$= \begin{bmatrix} x-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & \cdots & -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) \\ -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & x-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & \cdots & -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) \\ -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & x-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & \cdots & -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & -\frac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) & \cdots & x-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right) \end{bmatrix}_{m\times m}$$

Then by Proposition 1.6, we have

$$det\left(xI_m - A - B\left(\frac{1}{x}I_n + \frac{1}{x^2}D\right)C\right)$$
$$= \left[x-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right)+(m-1)\left(-\tfrac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right)\right)\right] \cdot \left[x-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right)-\left(-\tfrac{1}{2}-n\left(\frac{1}{x}+\frac{n-1}{2x^2}\right)\right)\right]^{m-1}$$
$$= \left(x-\frac{m-1}{2}-\frac{mn}{x}-\frac{mn(n-1)}{2x^2}\right)\cdot\left(x+\frac{1}{2}\right)^{m-1} \tag{3}$$

And similarly, $det(xI_n - D) = \begin{bmatrix} x & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & x & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & x & \cdots & -\frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & x \end{bmatrix}_{n\times n}$

$$= \left(x-\frac{n-1}{2}\right)\cdot\left(x+\frac{1}{2}\right)^{n-1} \tag{4}$$

Hence, put the value of equations (3) and (4) in the equation (2) and we get

$\varphi_H(K_{m,n};x)$

$= det(xI_n - D)\cdot det((xI_m - A) - B(xI_n - D)^{-1}C)$

$= \left(x - \frac{n-1}{2}\right)\cdot\left(x+\frac{1}{2}\right)^{n-1}\cdot\left(x - \frac{m-1}{2} - \frac{mn}{x} - \frac{mn(n-1)}{2x^2}\right)\cdot\left(x+\frac{1}{2}\right)^{m-1}$

$\Rightarrow \varphi_H(K_{m,n};x) = \left(x - \frac{n-1}{2}\right)\cdot\left(x+\frac{1}{2}\right)^{m+n-2}\cdot\left(x - \frac{m-1}{2} - \frac{mn}{x} - \frac{mn(n-1)}{2x^2}\right)$

Theorem 2.3. *Let G be a regular graph of order n with diameter atmost two and $h_1, h_2, h_3, ..., h_n$ be Harary eigenvalues of a graph G with $|h_i| \geq \frac{m-1}{2m}$, for all $1 \leq i \leq n$ then*

$$\mathcal{E}_H(D_m(G)) = m\mathcal{E}_H(G) + \frac{m-1}{2}\theta + mn - n$$

where, θ is the difference between the number of positive and negative Harary eigenvalue of the graph G.

Proof. Let G be a regular graph with vertex set $\{v_1, v_2, ..., v_n\}$ then the Harary matrix $H(G)$ is defined as

$$H(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array}\begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & \frac{1}{d(v_1,v_2)} & \frac{1}{d(v_1,v_3)} & \cdots & \frac{1}{d(v_1,v_n)} \\ \frac{1}{d(v_2,v_1)} & 0 & \frac{1}{d(v_2,v_3)} & \cdots & \frac{1}{d(v_2,v_n)} \\ \frac{1}{d(v_3,v_1)} & \frac{1}{d(v_3,v_2)} & 0 & \cdots & \frac{1}{d(v_2,v_n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{d(v_n,v_1)} & \frac{1}{d(v_n,v_2)} & \frac{1}{d(v_n,v_3)} & \cdots & 0 \end{bmatrix} \end{array}$$

Now, consider the m-copies $G_1, G_2, ..., G_m$ of a graph G and join each vertex u of graph G_i to the neighbors of the corresponding vertex v in the graph G_j, for all $1 \leq i, j \leq m$ to obtain $D_m(G)$. Then the Harary matrix of the graph $D_m(G)$ is defined as

$$H(D_m(G)) = \begin{bmatrix} H(G) & H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & \cdots & H(G)+\frac{1}{2}I_n \\ H(G)+\frac{1}{2}I_n & H(G) & H(G)+\frac{1}{2}I_n & \cdots & H(G)+\frac{1}{2}I_n \\ H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & H(G) & \cdots & H(G)+\frac{1}{2}I_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & \cdots & H(G) \end{bmatrix}_{mn\times mn}$$

$$\Rightarrow H(D_m(G))+\frac{1}{2}I_{mn} = \begin{bmatrix} H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & \cdots & H(G)+\frac{1}{2}I_n \\ H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & \cdots & H(G)+\frac{1}{2}I_n \\ H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & \cdots & H(G)+\frac{1}{2}I_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & H(G)+\frac{1}{2}I_n & \cdots & H(G)+\frac{1}{2}I_n \end{bmatrix}_{mn\times mn}$$

$\Rightarrow H(D_m(G)) + \frac{1}{2}I_{mn} = J_m \otimes [H(G) + \frac{1}{2}I_n]$

where, J_m is a matrix of order m with all the entries are 1. Moreover, $spec(J_m) = \begin{pmatrix} m & 0 \\ 1 & m-1 \end{pmatrix}$. Also, the eigenvalues of $H(G) + \frac{1}{2}I_n$ are $h_i + \frac{1}{2}$, for all $1 \le i \le n$.

Thus by Proposition 1.7, we get

$$spec(H(D_m(G)) + \tfrac{1}{2}I_{mn}) = \begin{pmatrix} m(h_i + \frac{1}{2}) & 0 \\ n & mn-n \end{pmatrix}$$

$$\Rightarrow spec(H(D_m(G))) = \begin{pmatrix} m(h_i + \frac{1}{2}) - \frac{1}{2} & -\frac{1}{2} \\ n & mn-n \end{pmatrix}$$

$$\Rightarrow spec(H(D_m(G))) = \begin{pmatrix} mh_i + \frac{m-1}{2} & -\frac{1}{2} \\ n & mn-n \end{pmatrix}$$

Now, as we have $|h_i| \ge \frac{m-1}{2m}$; for all $1 \le i \le n$,

$$\left|h_i + \frac{m-1}{2m}\right| = \begin{cases} |h_i| + \dfrac{m-1}{2m} & ;\ \text{for } h_i \geq 0 \\ |h_i| - \dfrac{m-1}{2m} & ;\ \text{for } h_i < 0 \end{cases}$$

Therefore,

$$\begin{aligned}
\mathcal{E}_H(D_m(G)) &= \sum_{i=1}^{n} \left|mh_i + \tfrac{1}{2}(m-1)\right| + \sum_{i=1}^{mn-n} |-1| \\
&= m\left(\sum_{i=1}^{n} \left|h_i + \tfrac{m-1}{2m}\right|\right) + (mn-n) \\
&= m\left(\sum_{h_i\geq 0} \left(|h_i| + \tfrac{m-1}{2m}\right) + \sum_{h_i<0} \left(|h_i| - \tfrac{m-1}{2m}\right)\right) + (mn-n) \\
&= m\left(\sum_{h_i\geq 0} |h_i| + \sum_{h_i<0} |h_i| + \tfrac{m-1}{2m}\left(\sum_{h_i\geq 0} 1 - \sum_{h_i<0} 1\right)\right) + mn - n \\
&= m\left(\mathcal{E}_H(G) + \tfrac{m-1}{2m}\theta\right) + mn - n;\ \text{as, } \theta \text{ is the difference between} \\
&\quad \text{the number of positive and negative Harary eigenvalue of graph } G \\
&= m\mathcal{E}_H(G) + \tfrac{m-1}{2}\theta + mn - n
\end{aligned}$$

Theorem 2.4. *Let G be a regular graph of order n with diameter atmost two and $h_1, h_2, ..., h_n$ be Harary eigenvalues of a graph G with $|h_i| \geq \dfrac{m-1}{m}$, for all $1 \leq i \leq n$ then*

$$\mathcal{E}_H(D^*_m(G)) = m\,\mathcal{E}_H(G) + (m-1)\theta + (mn-n)$$

where, θ is the difference between the number of positive and negative Harary eigenvalues of the graph G.

Proof. Let G be a regular graph with diameter atmost two and $H(G)$ be the Harary matrix of the graph G.

Now, consider the m-copies $G_1, G_2, ..., G_m$ of graph G and join each vertex u of a graph G_i to the neighbors of the corresponding vertex v and also with v in graph G_j for all $1 \leq i, j \leq m$ to obtain extended m-shadow $D^*_m(G)$. Then the Harary matrix of the graph $D^*_m(G)$ is given as

$$H(D^*_m(G)) = \begin{bmatrix} H(G) & H(G)+I_n & \cdots & H(G)+I_n \\ H(G)+I_n & H(G) & \cdots & H(G)+I_n \\ \vdots & \vdots & \ddots & \vdots \\ H(G)+I_n & H(G)+I_n & \cdots & H(G) \end{bmatrix}_{mn\times mn}$$

$$\Rightarrow H(D_m^*(G)) + I_{mn} = \begin{bmatrix} H(G)+I_n & H(G)+I_n & \cdots & H(G)+I_n \\ H(G)+I_n & H(G)+I_n & \cdots & H(G)+I_n \\ \vdots & \vdots & \ddots & \vdots \\ H(G)+I_n & H(G)+I_n & \cdots & H(G)+I_n \end{bmatrix}_{mn\times mn}$$
$$= J_m \otimes [H(G)+I_n]$$

where J_m is a matrix of order m with all the entries are 1. Moreover, $spec(J_m) = \begin{pmatrix} m & 0 \\ 1 & m-1 \end{pmatrix}$. Also, the eigenvalues of the matrix $H(G)+I_n$ are h_i+1, for all $1 \le i \le n$.

Hence, from Proposition 1.7, we get

$$spec\,(H(D_m^*(G)) + I_{mn}) = \begin{pmatrix} m(h_i+1) & 0 \\ n & mn-n \end{pmatrix}.$$

$$\Rightarrow spec\,(H(D_m^*(G))) = \begin{pmatrix} mh_i+(m-1) & -1 \\ n & mn-n \end{pmatrix}$$

Now, as we have $|h_i| \ge \frac{m-1}{m}$ for all $1 \le i \le n$,

$$\left|h_i + \frac{m-1}{m}\right| = \begin{cases} |h_i| + \dfrac{m-1}{m} & ;\ \text{for } h_i \ge 0 \\ |h_i| - \dfrac{m-1}{m} & ;\ \text{for } h_i < 0 \end{cases}$$

Hence,

$$\begin{aligned} \mathcal{E}_H(D_m^*(G)) &= \sum_{i=1}^{n} |mh_i + (m-1)| + \sum_{i=1}^{mn-n} |-1| \\ &= m\sum_{i=1}^{n} \left|h_i + \frac{m-1}{m}\right| + (mn-n) \\ &= m\left(\sum_{h_i\ge 0}\left(|h_i| + \frac{m-1}{m}\right) + \sum_{h_i<0}\left(|h_i| - \frac{m-1}{m}\right)\right) + (mn-n) \\ &= m\left(\sum_{h_i\ge 0}|h_i| + \sum_{h_i<0}|h_i| + \frac{m-1}{m}\left(\sum_{h_i\ge 0}1 - \sum_{h_i<0}1\right)\right) + (mn-n) \end{aligned}$$

$$\mathcal{E}_H(D_m^*(G)) = m\left(\mathcal{E}_H(G) + \frac{m-1}{m}\theta\right) + (mn-n);\ \text{as, } \theta \text{ is the difference between the number of positive and negative Harary } G \text{ eigenvalue of graph}$$
$$= m\,\mathcal{E}_H(G) + (m-1)\theta + (mn-n)$$

3. Conclusion

The energy of a graph is one of the important idea of spectral graph theory. This idea is a bond between chemical science and mathematical science. In this paper, I have derived a Harary characteristic polynomial of $K_{m,n}$ and the Harary energy of some graphs.

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